J. Fluid Mech. (1971), *vol.* 48, *part* 2, *pp.* 385–395
Printed in Great Britain

Third-harmonic resonance in the interaction of capillary and gravity waves

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(Received 23 September 1970)

The method of multiple scales is used to determine the temporal and spatial variation of the amplitudes and phases of capillary-gravity waves in a deep liquid at or near the third-harmonic resonant wave-number. This case corresponds to a wavelength of 2·99 cm in deep water. The temporal variation shows that the motion is always bounded, and the general motion is an aperiodic travelling wave. The analysis shows that pure amplitude-modulated waves are not possible in this case contrary to the second-harmonic resonant case. Moreover, pure phase-modulated waves are periodic even near resonance because the non-linearity adjusts the phases to yield perfect resonance. These periodic waves are found to be unstable, in the sense that any disturbance would change them into aperiodic waves.

1. Introduction

Resonances in the interaction of capillary and gravity waves were found by Harrison (1909) and Wilton (1915) to occur at the denumerable set of critical wave-numbers, $k'_n = (\rho g/nT)^{\frac{1}{2}}$, where n is an integer greater than unity, g is the body acceleration acting toward the liquid, and ρ and T are, respectively, the density and surface tension of the liquid. They found that the higher-order terms in a Stokes-type perturbation expansion are singular at these critical wave-numbers. The first two critical wave-numbers correspond to wavelengths of 2·44 and 2·99 cm in deep water.

Wilton modified his expansion so that the first term includes the fundamental and its second harmonic, and obtained a definite expansion for periodic travelling waves at the first critical wave-number. He found that two types of periodic waves could exist at this critical wave-number: one is capillary-like with a wave speed that decreases as the amplitude increases, and the other is gravity-like with a wave speed that increases as the amplitude increases. At this critical wave-number Wilton (1915) predicted single- and double-dimpled wave profiles, while Schooley (1960) observed double-dimpled wave profiles by means of enlarged pictures of short-fetch, wind-generated waves. Pierson & Fife (1961) determined a first-order expansion at or near the first critical wave-number using the method of straining of co-ordinates (Van Dyke 1964).

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Kamesvara Rav (1920) analyzed the non-linear capillary–gravity wave interaction in a liquid of finite depth. He found that the second approximation is singular for some wave-number. He produced, experimentally, wave profiles near this critical wave-number which show pronounced second-harmonic distortion. Barakat & Houston (1968) extended the analysis of Pierson & Fife to the case of a liquid with finite depth. Nayfeh (1970*a*) obtained a second-order expansion for periodic travelling waves at or near the first critical wave-number in a liquid of finite depth using the method of multiple scales (Nayfeh 1965*b*, 1968). Nayfeh & Saric (1971) extended the latter analysis to take into account the non-linear pressure perturbation exerted by a subsonic external flow on the surface of the liquid due to the appearance of waves.

McGoldrick (1965) analyzed the temporal resonant interactions of capillary and gravity waves within the framework of triad resonance (Phillips 1960; Benney 1962). Simmons (1969) and McGoldrick (1970*b*) investigated the temporal and spatial variation of the amplitudes and phases at the first-critical wave-number in a deep liquid using an averaging of the Lagrangian and the method of multiple scales, respectively. They found that three types of motion are possible: (i) pure amplitude modulation, (ii) pure phase modulation, and (iii) amplitude and phase modulation. McGoldrick (1970*a*) confirmed experimentally the theoretical results for pure amplitude-modulated waves. Nayfeh (1971*b*) extended the analysis of McGoldrick (1970*b*) by including the effects of (*a*) near resonance, (*b*) a finite liquid depth, and (*c*) pressure perturbations exerted by an external subsonic gas on the liquid surface. He found that the motion may be unbounded for certain gas-flow conditions, as in the cases of two-to-one (Nayfeh 1971*a*) and three-to-one (Nayfeh & Kamel 1970) resonances near the equilateral points in the restricted problem of three bodies where the motion may be unbounded. However, in the absence of the external gas, the motion is always bounded. Pure amplitude-modulated waves were found to be possible only at perfect resonance, and pure phase-modulated waves were found to be periodic due to the adjustment of the phases to yield perfect resonance.

Nayfeh (1970*b*) obtained a second-order expansion for the periodic waves near the second critical wave-number in a deep liquid using the method of multiple scales. He found that three periodic waves are possible near resonance; one is a gravity-like and the other two are capillary-like.

The purpose of the present paper is to investigate the temporal and spatial variation of the amplitudes and phases near the second critical wave-number using the method of multiple scales (Nayfeh 1965*a*). The next section contains the problem formulation. The non-resonant case is analysed in § 3 while the resonant case is analyzed in § 4.

2. Problem formulation

With respect to a Cartesian co-ordinate system whose x axis lies in the undisturbed free surface and its y axis normal to this surface and directed away from the liquid, the dimensionless potential function $\phi(x, y, t)$ representing the liquid motion is governed by

$$\nabla^2\phi = 0, \tag{2.1}$$

for
$$-\infty < x < \infty \quad \text{and} \quad -\infty < y \leqslant \eta(x,t), \tag{2.2}$$
where η is the elevation of the wave above the undisturbed surface. Here, distances and time are made dimensionless using the wave-number k' of the fundamental mode and the time $(gk')^{-\frac{1}{2}}$. The boundary conditions are
$$\nabla\phi \to 0 \quad \text{as} \quad y \to -\infty, \tag{2.3}$$
$$\eta_t + \phi_y = \eta_x \phi_x \quad \text{at} \quad y = \eta, \tag{2.4}$$
$$\eta - \phi_t = k^2 \eta_{xx}(1+\eta_x^2)^{-\frac{3}{2}} - \tfrac{1}{2}(\phi_x^2 + \phi_y^2) \quad \text{at} \quad y = \eta, \tag{2.5}$$
where
$$k = k'(T/\rho g)^{\frac{1}{2}}. \tag{2.6}$$

To determine an approximate solution to (2.1)–(2.5) for small but finite ϵ (i.e. maximum wave steepness ratio), we use the method of multiple scales (Nayfeh 1965*a*, 1968) by introducing the temporal scales $T_0 = t$ and $T_2 = \epsilon^2 t$, and the spatial scales $X_0 = x$ and $X_2 = \epsilon^2 x$. Moreover, we assume that η and ϕ possess uniformly valid expansions of the form
$$\eta = \sum_{n=1}^{3} \epsilon^n \eta_n(X_0, X_2, T_0, T_2) + O(\epsilon^4), \tag{2.7}$$
$$\phi = \sum_{n=1}^{3} \epsilon^n \phi_n(X_0, X_2, y, T_0, T_2) + O(\epsilon^4). \tag{2.8}$$
The functions η and ϕ were taken to be independent of the scales X_1 and T_1 because the resonant interactions do not occur before $O(\epsilon^3)$. Had we included their effects, we would have found that η and ϕ are independent of them.

Substituting (2.7) and (2.8) into (2.1)–(2.5), and equating coefficients of like powers of ϵ, we find that each ϕ_n satisfies (2.3), and

order ϵ
$$\frac{\partial^2\phi_1}{\partial X_0^2} + \frac{\partial^2\phi_1}{\partial y^2} = 0, \tag{2.9}$$
$$\frac{\partial\eta_1}{\partial T_0} + \frac{\partial\phi_1}{\partial y} = 0 \quad \text{at} \quad y = 0, \tag{2.10}$$
$$\eta_1 - \frac{\partial\phi_1}{\partial T_0} - k^2\frac{\partial^2\eta_1}{\partial X_0^2} = 0 \quad \text{at} \quad y = 0; \tag{2.11}$$

order ϵ^2
$$\frac{\partial^2\phi_2}{\partial X_0^2} + \frac{\partial^2\phi_2}{\partial y^2} = 0, \tag{2.12}$$
$$\frac{\partial\eta_2}{\partial T_0} + \frac{\partial\phi_2}{\partial y} = \frac{\partial\phi_1}{\partial X_0}\frac{\partial\eta_1}{\partial X_0} - \eta_1\frac{\partial^2\phi_1}{\partial y^2} \quad \text{at} \quad y = 0, \tag{2.13}$$
$$\eta_2 - \frac{\partial\phi_2}{\partial T_0} - k^2\frac{\partial^2\eta_2}{\partial X_0^2} = \eta_1\frac{\partial^2\phi_1}{\partial y\,\partial T_0} - \frac{1}{2}\left(\frac{\partial\phi_1}{\partial X_0}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi_1}{\partial y}\right)^2 \quad \text{at} \quad y = 0; \tag{2.14}$$

order ϵ^3
$$\frac{\partial^2\phi_3}{\partial X_0^2} + \frac{\partial^2\phi_3}{\partial y^2} = -2\frac{\partial^2\phi_1}{\partial X_0\,\partial X_2}, \tag{2.15}$$
$$\frac{\partial\eta_3}{\partial T_0} + \frac{\partial\phi_3}{\partial y} = \frac{\partial\phi_1}{\partial X_0}\frac{\partial\eta_2}{\partial X_0} + \left(\frac{\partial^2\phi_1}{\partial X_0\,\partial y}\eta_1 + \frac{\partial\phi_2}{\partial X_0}\right)\frac{\partial\eta_1}{\partial X_0} - \eta_1\frac{\partial^2\phi_2}{\partial y^2}$$
$$- \tfrac{1}{2}\eta_1^2\frac{\partial^3\phi_1}{\partial y^3} - \eta_2\frac{\partial^2\phi_1}{\partial y^2} - \frac{\partial\eta_1}{\partial T_2} \quad \text{at} \quad y = 0, \tag{2.16}$$

$$\eta_3-\frac{\partial\phi_3}{\partial T_0}-k^2\frac{\partial^2\eta_3}{\partial X_0^2} = -\frac{\partial\phi_1}{\partial X_0}\frac{\partial\phi_2}{\partial X_0}-\frac{\partial\phi_1}{\partial y}\frac{\partial\phi_2}{\partial y}$$
$$-\eta_1\left(\frac{\partial^2\phi_1}{\partial X_0\partial y}\frac{\partial\phi_1}{\partial X_0}+\frac{\partial^2\phi_1}{\partial y^2}\frac{\partial\phi_1}{\partial y}\right)-\tfrac{3}{2}k^2\frac{\partial^2\eta_1}{\partial X_0^2}\left(\frac{\partial\eta_1}{\partial X_0}\right)^2$$
$$+\eta_1\frac{\partial^2\phi_2}{\partial y\partial T_0}+\tfrac{1}{2}\eta_1^2\frac{\partial^3\phi_1}{\partial y^2\partial T_0}+\eta_2\frac{\partial^2\phi_1}{\partial y\partial T_0}$$
$$+\frac{\partial\phi_1}{\partial T_2}+2k^2\frac{\partial^2\eta_1}{\partial X_0\partial X_2}\quad\text{at}\quad y=0. \tag{2.17}$$

The solution of the first-order problem is

$$\eta_1 = \sum_{n=1}^{\infty}\epsilon^n[A_n(X_2,T_2)e^{i\theta_n(X_0,T_0)}+\bar{A}_n(X_2,T_2)e^{-i\theta_n(X_0,T_0)}], \tag{2.18}$$

$$\phi_1 = -i\sum_{n=1}^{\infty}\frac{\mu_n}{n}[A_n(X_2,T_2)e^{i\theta_n(X_0,T_0)}-\bar{A}_n(X_2,T_2)e^{-\theta_n(X_0,T_0)}]\,e^{ny}, \tag{2.19}$$

where
$$\theta_n = nX_0+\mu_nT_0,\quad \mu_n^2 = n(n^2k^2+1). \tag{2.20}$$

The resonant wave-numbers are given by

$$\mu_n = n\mu_1 \quad\text{or}\quad k_n^2 = 1/n. \tag{2.21}$$

The non-resonant case is treated in the next section while the resonant case with $n = 3$ is treated in §4.

3. Non-resonant case

In this case, we take

$$\eta_1 = A_1(X_2,T_2)\,e^{i\theta_1}+\bar{A}_1(X_2,T_2)\,e^{-i\theta_1}, \tag{3.1a}$$

$$\phi_1 = -i\mu_1[A_1(X_2,T_2)\,e^{i\theta_1}-\bar{A}_1(x_2,T_2)e^{-\theta_1}]\,e^y. \tag{3.1b}$$

Substituting for η_1 and ϕ_1 into the second-order equations, and solving for η_2 and ϕ_2, we get

$$\eta_2 = \frac{1+k^2}{1-2k^2}[A_1^2e^{2i\theta_1}+\bar{A}_1^2e^{-2i\theta_1}], \tag{3.2}$$

$$\phi_2 = -3i\mu_1\frac{k^2}{1-2k^2}[A_1^2e^{2i\theta_1}-\bar{A}_1^2e^{-2i\theta_1}]\,e^{2y}. \tag{3.3}$$

With the first- and second-order solutions known, (2.15)–(2.17) become

$$\frac{\partial^2\phi_3}{\partial X_0^2}+\frac{\partial^2\phi_3}{\partial y^2} = -2\mu_1\frac{\partial A_1}{\partial X_2}e^{i\theta_1}e^y+\mathrm{CC}, \tag{3.4}$$

$$\frac{\partial\eta_3}{\partial T_0}+\frac{\partial\phi_3}{\partial y} = \left[-\frac{\partial A_1}{\partial T_2}+\tfrac{1}{2}i\mu_1\frac{5+8k^2}{1-2k^2}A_1^2A_1\right]e^{i\theta_1}$$
$$+\tfrac{9}{2}i\mu_1\frac{1+4k^2}{1-2k^2}A_1^3e^{3i\theta_1}+\mathrm{CC}, \tag{3.5}$$

$$\eta_3-\frac{\partial\phi_0}{\partial T_0}-k^2\frac{\partial^2\eta_3}{\partial X_0^2} = \left[-i\mu_1\frac{\partial A_1}{\partial T_2}+2ik^2\frac{\partial A_1}{\partial X_2}-\mu_1^2\left(\frac{5}{2}-\frac{1+k^2}{1+2k^2}-\frac{3}{2}\frac{k^2}{1+k^2}\right)A_1^2\bar{A}_1\right]e^{i\theta_1}$$
$$+\mu_1^2\left(\frac{1}{2}-\frac{3}{2}\frac{k^2}{1+k^2}+\frac{1+13k^2}{1-2k^2}\right)A_1^3e^{3i\theta_1}+\mathrm{CC}, \tag{3.6}$$

where CC stands for the complex conjugate. The particular solution of (3.4)–(3.6) subject to $\nabla\phi_3 \to 0$ as $y \to -\infty$ contains secular terms of the form T_0 or $X_0 \exp i\theta_1$ which make η_3/η_1 be unbounded as T_0 or $X_0 \to \infty$. The condition which must be satisfied for there to be no secular terms is

$$\frac{\partial A_1}{\partial T_2} - u_1 \frac{\partial A_1}{\partial X_2} = \tfrac{1}{4} i\mu_1 \frac{8+k^2+2k^4}{(1-2k^2)(1+k^2)} A_1^2 \bar{A}_1, \tag{3.7}$$

where $u_1 = (3k^2+1)/2\mu_1$ is the dimensionless group velocity.

Letting $A_1 = \frac{1}{2} a_1 \exp i\beta_1$ with real a_1 and β_1 and separating real and imaginary parts in (3.7) we get

$$\frac{\partial a_1}{\partial T_2} - u_1 \frac{\partial a_1}{\partial X_2} = 0, \tag{3.8}$$

$$\frac{\partial \beta_1}{\partial T_2} - u_1 \frac{\partial \beta_1}{\partial X_2} = \tfrac{1}{16} \mu_1 \frac{8+k^2+2k^4}{(1-2k^2)(1+k^2)} a_1^2. \tag{3.9}$$

The solution of (3.8) is

$$a_1 = f(X_2 + u_1 T_2), \tag{3.10}$$

where the function f is determined from the initial conditions. Then, the solution of (3.9) is

$$\beta_1 = -\frac{1}{32} \frac{\mu_1}{u_1} \frac{8+k^2+2k^4}{(1-2k^2)(1+k^2)} [(X_2 - u_1 T_2) f^2(X_2 + u_1 T_2) + g(X_2 - u_1 T_2)], \tag{3.11}$$

where g is also determined from the initial conditions. Thus, the motion consists of amplitude- and phase-modulated waves.

The free surface to third order is

$$\eta = \epsilon a_1 \cos\theta_1 + \tfrac{1}{2}\epsilon \frac{1+k^2}{1-2k^2} a_1^2 \cos 2\theta_1 + \tfrac{3}{16}\epsilon^3 \frac{2k^4+7k^2+2}{(1-2k^2)(1-3k^2)} a_1^3 \cos 3\theta_1 + O(\epsilon^4), \tag{3.12}$$

where

$$a_1 = f(\epsilon^2 x + \epsilon^2 u_1 t), \tag{3.13}$$

$$\theta_1 = x + (1+k^2)^{\frac{1}{2}} t - \frac{1}{32} \frac{\mu_1}{u_1} \frac{8+k^2+2k^4}{(1-2k^2)(1+k^2)} \times [\epsilon^2 (x - u_1 t) f^2(\epsilon^2 x + \epsilon^2 u_1 t) + g(\epsilon^2 x - \epsilon^2 u_1 t)] + O(\epsilon^3). \tag{3.14}$$

If the initial conditions are such that the spatial variation of a_1 and β_1 vanishes, then $a_1 =$ constant, and

$$\beta_1 = \tfrac{1}{16} \mu_1 \frac{8+k^2+2k^4}{(1-2k^2)(1+k^2)} a_1^2 T_2 + \text{const.} \tag{3.15}$$

Then, the free surface is still given by (3.12) but

$$\theta_1 = x + (k^2+1)^{\frac{1}{2}} \left[1 + \tfrac{1}{16}\epsilon^2 \frac{8+k^2+2k^4}{(1-2k^2)(1+k^2)}\right] t + O(\epsilon^3). \tag{3.16}$$

This solution is in agreement with those of Pierson & Fife (1961) and Nayfeh (1970*b*).

Equation (3.12) shows that $\eta \to \infty$ as $k^2 = \frac{1}{2}$ or $\frac{1}{3}$. These are the first two resonant wave-numbers. The first resonant case is analyzed by Simmons (1969), McGoldrick (1970*b*), and Nayfeh (1971*b*). The second resonant case is analyzed in the next section.

4. Resonant case

In this case we assume that

$$\mu_3 - 3\mu_1 = \epsilon^2\sigma, \tag{4.1a}$$

where the determining $\sigma = O(1)$. Since $\mu^2 = n(n^2k^2+1)$, (4.1a) gives

$$k^2 = \frac{1}{3} + \frac{2\surd 3}{9}\epsilon^2\sigma + \dots. \tag{4.1b}$$

With this expression for k^2, (2.9)–(2.17) remain unchanged except k^2 in (2.11), (2.14), and (2.17) is replaced by $\frac{1}{3}$, and the right-hand side of (2.17) is augmented by the term $(2\surd 3/9)\sigma(\partial^2\eta_1/\partial X_0^2)$.

The solution of the first-order problem is taken to be

$$\eta_1 = \sum_{n=1,3}[A_n(X_2, T_2)e^{in\theta} + \bar{A}_n(X_2, T_2)e^{-in\theta}], \tag{4.2}$$

$$\phi_1 = -i\mu_1\sum_{n=1,3}[A_n(X_1, T_2)e^{in\theta} - \bar{A}_n(X_2, T_2)e^{in\theta}], \tag{4.3}$$

where
$$\theta = X_0 + \mu_1 T_0 \quad (\mu_1 = 2\surd(3)/3). \tag{4.4}$$

Substituting for η_1 and ϕ_1 into the second-order equations, and solving for η_2 and ϕ_2, we get

$$\eta_2 = 4(A_1^2 + \bar{A}_1A_3)e^{2i\theta} - 8A_1A_3e^{4i\theta} - \tfrac{12}{5}A_3e^{6i\theta} + \mathrm{CC}, \tag{4.5}$$

$$\begin{aligned}\phi_2 = {} & -3i\mu_1(A_1^2 + \bar{A}_1A_3)e^{2y}e^{2i\theta} + 12i\mu_1A_1A_3e^{4y}e^{4i\theta} \\ & + \tfrac{27}{5}i\mu_1A_3^2e^{6y}e^{6i\theta} + \mathrm{CC}.\end{aligned} \tag{4.6}$$

With substitution of the first- and second-order solutions into (2.15), (2.16) and the modified (2.17) we obtain

$$\frac{\partial^2\phi_3}{\partial X_0^2} + \frac{\partial^2\phi_3}{\partial y^2} = -2\mu_1\frac{\partial A_1}{\partial X_2}e^{i\theta}e^{y} - 6\mu_1\frac{\partial A_2}{\partial X_2}e^{3i\theta}e^{3y} + \mathrm{CC}, \tag{4.7}$$

$$\begin{aligned}\frac{\partial\eta_3}{\partial T_0} + \frac{\partial\phi_3}{\partial y} = {} & \left[\tfrac{1}{2}i\mu_1(23A_1^2\bar{A}_1 + 127\bar{A}_1^2A_3 + 38A_1A_3\bar{A}_3) - \frac{\partial A_1}{\partial T_2}\right]e^{i\theta} \\ & + \left[\tfrac{1}{10}i\mu_1(315A_1^3 - 150\bar{A}_1A_1A_3 - 783A_3^2\bar{A}_3) - \frac{\partial A_3}{\partial T_2}\right]e^{3i\theta} \\ & - \tfrac{1}{2}i\mu_1(325A_1^2 - 467\bar{A}_1A_3)A_3e^{5i\theta} - \tfrac{6811}{10}i\mu_1A_1A_3^2e^{7i\theta} \\ & - \tfrac{3159}{10}i\mu_1A_3^3e^{9i\theta} + \mathrm{CC},\end{aligned} \tag{4.8}$$

$$\begin{aligned}& \eta_3 - \frac{\partial\phi_3}{\partial T_0} - \frac{1}{3}\frac{\partial^2\eta_3}{\partial X_0^2} \\ & = \left[\tfrac{1}{8}\mu_1^2(15A_1^2\bar{A}_1 + 139\bar{A}_1^2A_3 + 14A_1A_3\bar{A}_3) - i\mu_1\frac{\partial A_1}{\partial T_2} - \frac{2\surd 3}{9}\sigma A_1 + \tfrac{2}{3}i\frac{\partial A_1}{\partial X_2}\right]e^{i\theta} \\ & + \left[\tfrac{1}{40}\mu_1^2(645A_1^3 - 1290A_1\bar{A}_1A_3 - 2349A_3^2\bar{A}_3) - i\mu_1\frac{\partial A_3}{\partial T_2} - 2\surd 3\sigma A_3 + 2i\frac{\partial A_3}{\partial X_2}\right]e^{3i\theta} \\ & - \tfrac{1}{8}\mu_1^2(1145A_1^2 - 643\bar{A}_1A_3)A_3e^{5i\theta} - \tfrac{18277}{40}\mu_1^2A_1A_3^2e^{7i\theta} - \tfrac{1863}{8}\mu_1^2A_3^3e^{9i\theta} + \mathrm{CC}.\end{aligned} \tag{4.9}$$

To determine the conditions which must be satisfied for there to be no secular terms in the solution of the third-order problem, we let

$$\eta_3 = 0, \quad \phi_3 = \left(B_1 - \mu_1 \frac{\partial A_1}{\partial X_2} y\right) e^{i\theta} e^{y} + \left(B_3 - \mu_1 \frac{\partial A_3}{\partial X_2} y\right) e^{3i\theta} e^{3y}. \tag{4.10}$$

Substituting into (4.8) and (4.9) and equating the coefficients of each of $e^{i\theta}$ and $e^{3i\theta}$ on both sides, we obtain

$$B_1 - \mu_1 \frac{\partial A_1}{\partial X_2} = \tfrac{1}{8} i\mu_1 (23 A_1^2 \bar{A}_1 + 127 \bar{A}_1^2 A_3 + 38 A_1 A_3 \bar{A}_3) - \frac{\partial A_1}{\partial T_2}, \tag{4.11}$$

$$-i\mu_1 B_1 = \tfrac{1}{8}\mu_1^2 (15 A_1^2 \bar{A}_1 + 139 \bar{A}_1^2 A_3 + 14 A_1 A_3 \bar{A}_3) - i\mu_1 \frac{\partial A_1}{\partial T_2} + \tfrac{2}{3} i \frac{\partial A_1}{\partial X_2} - \frac{2\sqrt{3}}{9} \sigma A_1, \tag{4.12}$$

$$3B_3 - \mu_1 \frac{\partial A_3}{\partial X_2} = \tfrac{1}{10} i\mu_1 (315 A_1^3 - 150 \bar{A}_1 A_1 A_3 - 783 A_3^2 A_3) - \frac{\partial A_3}{\partial T_2}, \tag{4.13}$$

$$-3i\mu_1 B_3 = \tfrac{1}{40}\mu_1^2 (645 A_1^3 - 1290 A_1 \bar{A}_1 A_3 - 2349 A_3^2 A_3) - i\mu_1 \frac{\partial A_3}{\partial T_2} + 2i \frac{\partial A_3}{\partial X_2} - 2\sqrt{3}\sigma A_3. \tag{4.14}$$

Elimination of B_1 and B_3 from (4.11)–(4.14) yields

$$\frac{\partial A_1}{\partial T_2} - u_1 \frac{\partial A_1}{\partial X_2} = \tfrac{1}{6} i\sigma A_1 + \tfrac{1}{16} i\mu_1 (77 A_1^2 \bar{A}_1 + 369 \bar{A}_1^2 A_3 + 138 A_1 A_3 \bar{A}_3), \tag{4.15}$$

$$\frac{\partial A_3}{\partial T_2} - u_3 \frac{\partial A_3}{\partial X_2} = \tfrac{3}{2} i\sigma A_3 + \tfrac{3}{80} i\mu_1 (205 A_1^3 + 230 A_1 \bar{A}_1 A_3 - 261 A_3^2 \bar{A}_3), \tag{4.16}$$

where $u_1 = \sqrt{(3)}/2$ and $u_3 = 5\sqrt{(3)}/6$ are the dimensionless group velocities of the two modes. With the secular terms eliminated, the solution for η_3 becomes

$$\eta_3 = C_1(X_2, T_2)\, e^{i\theta} + C_3(X_2, T_2)\, e^{3i\theta} + \tfrac{1}{16}(155 A_1^2 A_3 - 1225 \bar{A}_1 A_3)\, e^{5i\theta} + \tfrac{2989}{80} A_1 A_3^2 e^{7i\theta} + \tfrac{1107}{160} A_3^3 e^{9i\theta} + \text{complex conjugate}, \tag{4.17}$$

where the functions C_1 and C_3 need to be determined by carrying out the expansion to fourth order. This is not done in this paper, and these functions remain undetermined.

To analyze the solutions of (4.15) and (4.16), we let $A_n = \frac{1}{2} a_n \exp i\beta_n$ with real a_n and β_n, separate real and imaginary parts and obtain

$$\frac{\partial a_1}{\partial T_2} - u_1 \frac{\partial a_1}{\partial X_2} = -\tfrac{369}{64} \mu_1 a_1^2 a_3 \sin\alpha, \tag{4.18}$$

$$\frac{\partial a_3}{\partial T_2} - u_3 \frac{\partial a_3}{\partial X_2} = \tfrac{123}{64} \mu_1 a_1^3 \sin\alpha, \tag{4.19}$$

$$a_1 \left(\frac{\partial \beta_1}{\partial T_2} - u_1 \frac{\partial \beta_1}{\partial X_2}\right) = \tfrac{1}{6}\sigma a_1 + \tfrac{1}{64}\mu_1 (77 a_1^3 + 369 a_1^2 a_3 \cos\alpha + 138 a_1 a_3^2), \tag{4.20}$$

$$a_3 \left(\frac{\partial \beta_3}{\partial T_2} - u_3 \frac{\partial \beta_3}{\partial X_2}\right) = \tfrac{3}{2}\sigma a_3 + \tfrac{3}{320}\mu_1 (305 a_1^3 \cos\alpha + 230 a_1^2 a_3 - 261 a_3^3), \tag{4.21}$$

where

$$\alpha = \beta_3 - 3\beta_1. \tag{4.22}$$

There is no general solution available yet for the solution of (4.18)–(4.21) subject to arbitrary initial conditions. In what follows, we assume that the initial conditions are such that the spatial variation of a_n and β_n vanishes. The resultant equations are similar in form to those obtained by Nayfeh & Kamel (1970) for the problem of three-to-one resonances near the equilateral librations points in the restricted problem of three bodies.

Using (4.22), we can combine (4.20) and (4.21) into

$$a_3 \frac{d\alpha}{dT_2} = \sigma a_3 - \frac{\sqrt{3}}{160}[951 a_3^3 + 1845 a_3^2 a_1 \cos\alpha + 155 a_3 a_1^2 - 205 a_1^3 \cos\alpha]. \quad (4.23)$$

Adding a_1 times (4.18) to $3a_3$ times (4.19), and integrating the resultant equation, we obtain

$$a_1^2 + 3a_3^2 = E, \quad (4.24)$$

where E is a constant representing the energy density of the system. Equation (4.24) shows that the liquid surface is always bounded in contrast with the three-to-one resonances near the equilateral libration points (Nayfeh & Kamel) where the motion may be unbounded. Elimination of T_2 from (4.18) and (4.23) yields

$$a_1^3 a_3 \frac{d\cos\alpha}{da_1} + \left(3a_1^2 a_3 - \frac{1}{3}\frac{a_1^4}{a_3}\right)\cos\alpha = \frac{32}{123\sqrt{3}}\sigma a_1 - \tfrac{317}{205} a_3^2 a_1 - \tfrac{31}{123} a_1^3. \quad (4.25)$$

Since $a_1 da_1 = -3a_3 da_3$ from (4.24), the solution of (4.25) is

$$a_1^3 a_3 \cos\alpha = \frac{32}{369}\frac{\sigma}{\mu_1} a_1^2 + \tfrac{951}{820} a_3^4 - \tfrac{31}{492} a_1^4 + L, \quad (4.26)$$

where L is another constant of integration. One more integral of the motion is needed to complete the description of the motion; this integral seems to be available by numerical integration only.

The description of the motion can be reduced to the solution of a first-order differential equation by letting

$$a_1^2 = E\xi \quad \text{and hence} \quad a_3^2 = \tfrac{1}{3}E(1-\xi). \quad (4.27)$$

Elimination of α, a_1^2, a_3^2 from (4.18), (4.26) and (4.27) yields

$$\left(\frac{16}{123E}\frac{d\xi}{dT_2}\right)^2 = F^2(\xi) - G^2(\xi), \quad (4.28)$$

where

$$F(\xi) = \pm[\xi^3(1-\xi)]^{\frac{1}{2}}, \quad (4.29)$$

$$G(\xi) = \sqrt{(3)}\left[-\tfrac{31}{492}\xi^2 + \tfrac{317}{2460}(1-\xi)^2 + \frac{32}{369}\frac{\sigma}{\mu_1 E}\xi + \frac{L}{E^2}\right]. \quad (4.30)$$

The functions $F(\xi)$ and $G(\xi)$ are shown schematically in figure 1. Since a_1, and hence ξ, must be real, F^2 must be greater than or equal to G^2. The points where G meets F correspond to the vanishing of $d\xi/dT_2$, and hence the vanishing of both da_1/dT_2 and da_3/dT_2. A curve such as G_2 which meets one branch of F at two different points or G_3 which meets both branches corresponds to a periodic solution for a_1, a_3 and α, and hence corresponds to an aperiodic wave.

On the other hand, a point such as P where G_1 touches F represents an equilibrium point (stationary solution for a_1 and a_3, and hence for α). Consequently, the motion corresponding to such a point is a periodic wave. Periodic waves correspond to $F' = G'$, which is equivalent to the stationary solutions of (4.18), (4.19) and (4.23). The stationary solutions of these equations are given by

$$\sin\alpha_0 = 0 \quad \text{or} \quad \alpha_0 = n\pi, \tag{4.31}$$

$$951a_{30}^3 + 1845a_{30}^2 a_{10}\cos n\pi + 155a_{30}a_{10}^2 - 205a_{10}^3\cos n\pi - (160/\surd 3)\,\sigma a_{30} = 0. \tag{4.32}$$

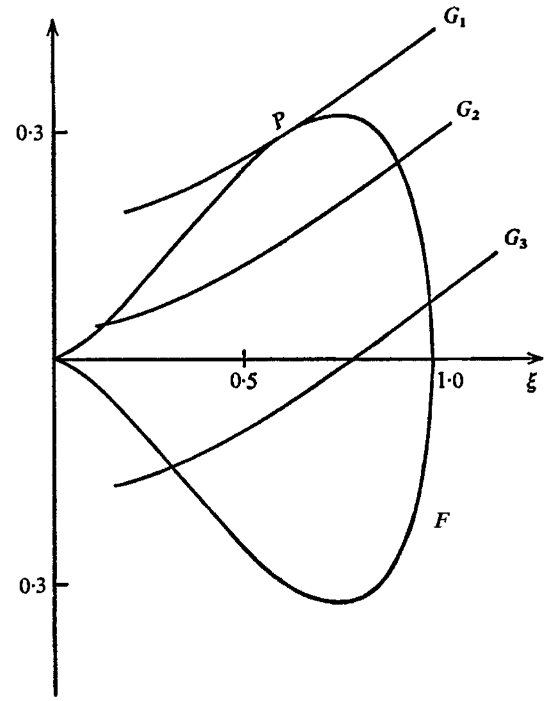

FIGURE 1. Topology of motion.

The periodic wave solution obtained by Nayfeh (1970*b*) corresponds to (4.32) with $n = 0$. Equation (4.32) is a cubic for a_{30} in terms of $a_{10}\cos n\pi$. In the case $n = 0$, Nayfeh (1970*b*) found that for any a_{10} and $\sigma > -5.835a_{10}^2$ there are three real roots for a_{30}/a_{10}. Two of these roots correspond to capillary-like waves while the third root corresponds to a gravity-like wave. Below $\sigma = -5{\cdot}835a_{10}^2$ there is only one real root for a_{30}/a_{10}, and hence one wave profile could exist. Figure 1 shows that any small disturbance would lead to a curve such as G_2 which meets one branch of F at two different points, and hence the subsequent motion is an aperiodic wave.

The motions discussed so far are pure phase-modulated waves (periodic waves), and amplitude- and phase-modulated aperiodic waves. Pure amplitude-modulated waves do not exist in this case because if β_1 and β_2 are constants, α is constant, hence (4.20), (4.21) and (4.24) give constant values for a_1 and a_2 in terms of the constants E and α.

Equations (4.18)–(4.21) show that there exists a periodic wave with $a_1 = 0$. In this case, $a_3 = \text{const.}$, and

$$\beta_3 = (\tfrac{3}{2}\sigma - \tfrac{783}{320}a_3^2 T_2 + \text{const}). \tag{4.33}$$

On the other hand, there is no solution in which $a_3 = 0$ while $a_1 \neq 0$.

5. Summary

The method of multiple scales is used to investigate three-to-one resonances (third-harmonic resonance) in the interaction of capillary and gravity waves in a deep liquid. Equations that govern the temporal as well as the spatial variation of the amplitudes and phases of the fundamental and its third harmonic are presented. Since there is no general solution available, yet, for these equations subject to arbitrary initial conditions, the temporal behaviour of the solution is investigated (the same results hold for the spatial variation).

The temporal variation of the amplitudes and the phases shows that the motion is always bounded even at perfect resonance as in the second-harmonic resonant case (Simmons 1969; McGoldrick 1970*b*; Nayfeh 1971*b*) and in contrast with the cases of two-to-one (Nayfeh 1971*a*) and three-to-one (Nayfeh & Kamel 1970) resonances near the equilateral libration points in the restricted problem of three bodies where the motion may be unbounded. Since the introduction of an external subsonic gas leads to instability in the second-harmonic resonance case for certain flow conditions (Nayfeh 1971*b*), it may lead to instability in this case. This still needs to be investigated.

The general motion is an aperiodic travelling wave. It is found that pure amplitude-modulated waves are not possible even at perfect resonance, contrary to the second-harmonic resonant case (Simons 1969; McGoldrick 1970*b*; Nayfeh 1971*b*). However, pure phase-modulated waves are possible even near resonance as in the second-harmonic resonant case, and they correspond to periodic waves. The non-linear motion adjusts the phases so that the wave speeds of the fundamental and its third harmonic are the same, thereby producing perfect resonance. It is found that these periodic waves are unstable, in the sense that any disturbance applied to one of them would lead to an aperiodic wave.

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